Proceedings of the Jangjeon Mathematical Society
21 (2018), No. 3. pp. 357 - 373
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2018.21.3.357

EDGE-ADJACENCY IN GRAPHS

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Abstract

Energy of a graph was defined by E. Hückel as the sum of absolute values of the eigenvalues of the adjacency matrix during his search to find a method to obtain approximate solutions of Schrödinger equation for a class of organic molecules. It is an important sub-area of graph theory called spectral graph theory. Schrödinger equation is a second order differential equation which includes the energy of the corresponding system and as we can model all molecules with graphs, we can calculate the energy of a given graph. Here we obtain the exact formulae and recurrence relations for the edge-characteristic and incidency polynomials of some well-known graph classes.

1 Introduction

[1] [2] [3]

Let $G = (V, E)$ be a simple connected graph, that is G is a graph with no loops nor multiple edges. Two vertices u and v of G are called adjacent if there is an edge e of G connecting u to v. If G has n vertices $v_1, v_2, \cdots, v_n$, we can form an $n \times n$ matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1, & if\ v_i\ and\ v_j\ are\ adjacent \\ 0, & otherwise. \end{cases}$$

This matrix is called the adjacency matrix of the graph G. The set of all eigenvalues of the adjacency matrix A is called the spectrum of the graph G, denoted by $S(G)$. These eigenvalues are

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[1]AMS 2010 Subject Classification Number: 05C31, 05C38, 05C50
[2]Keywords: Graph polynomials, edge adjacency, incidency, energy
[3]This work was supported by the research fund of Uludag University project no: F-2015/17

also called the eigenvalues of the graph G.

As well-known, the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of a square $n \times n$ matrix A are the roots of the equation $|A - \lambda I_n| = 0$. The polynomial on the left hand side of this equation is called the characteristic polynomial of A (and of the graph G with a slight abuse of language). This polynomial is sometimes called as the spectral polynomial of G.

The sum of absolute values of the eigenvalues of G is called the energy of G, which is an important aspect for the subfield of graph theory called spectral graph theory, see [1], [5], [3], [6], [8], [9], [10].

As usual, we denote path, cycle, star, complete and complete bipartite graphs by P_n, C_n, S_n, K_n and $K_{r,s}$, respectively.

The spectrum of some graph types including path, cycle, star, complete and complete bipartite graphs are known in literature, [4], [3], [2]. The spectrum of path and cycle graphs show differences with the other graph types as they can be stated in terms of roots of unity. The spectrum of these graph types by means of the characteristic polynomial are obtained in detail in [4].

Similarly to the notion of vertex-adjacency, we shall study the parallel notions of edge-adjacency and incidency. We shall give exact formulae for the edge-characteristic polynomials and also some recurrence relations. We find that the sum of the coefficients of the edge-characteristic polynomials of the path and cycle graphs are obtained in modulo 6. We also give some relations between the vertex-characteristic and edge-characteristic polynomials. . We finally obtain some incidency polynomials which do not exists always.

An important and obvious relation between the edge-adjacency and vertex adjacency is given by means of the line graph $L(G)$ of G as follows, [7]:

Lemma 1.1. *The edge-adjacency matrix of a graph G is identical to the vertex-adjacency matrix of the corresponding line graph $L(G)$ of G. That is,*

$$A^e(G) = A^v(L(G)).$$

2 Edge-adjacency matrices and polynomials

Let G be a simple connected graph having n vertices and m edges. The (vertex) adjacency matrix A of G is a square $n \times n$ matrix A $= [a_{ij}]_{n\times n}$ determined by the adjacency of vertices as follows:

$$a_{ij} = \begin{cases} 1, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

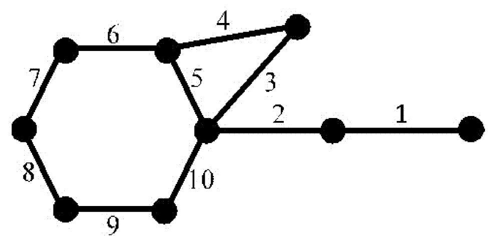

Figure 1: A graph G

The adjacency matrix is used in many areas of graph theory and also in molecular chemistry. The sum of the absolute values of the eigenvalues of the adjacency matrix gives the energy of a graph G. Therefore the adjacency matrix and related notions for adjacency are frequently used in many areas. In this paper we shall study some similar matrices corresponding to graphs.

The edge-adjacency matrix A^e of G is a square $m \times m$ matrix $\mathrm{A}^e = [a_{ij}]_{m\times m}$ determined by the adjacency of edges as follows:

$$a_{ij} = \begin{cases} 1, & \text{if the edges } e_i \text{ and } e_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathrm{A}^e = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

In Figure 1, there is a graph G with 10 labelled edges and its edge-adjaceny matrix is given above. Although the vertex-adjacency matrix of non-isomorphic graphs must be different, edge-adjacency matrices of non-isomorphic graphs could be the same. For example the graphs in Figure 2 both have the same edge adjacency matrix.

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

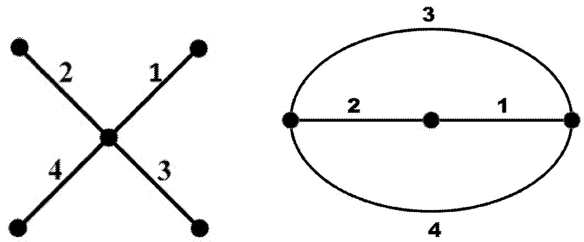

Figure 2: Non-isomorphic graphs having same edge-adjacency matrix

In the literature, when one mentions the characteristic equation corresponding to a graph G, it is commonly understood that one is dealing with the adjacency matrix $A(G)$. As we shall study the characteristic polynomials corresponding to the edge-adjacency matrix A^e, we shall denote the edge-adjacency polynomial obtained by calculating the characteristic polynomial of the edge-adjacency matrix by $P^e_G(\lambda)$ to differ them from vertex-adjacency polynomials. First we start with the path graph P_n. We have

Theorem 2.1. *The recurrence relation for the edge-characteristic polynomial of the path graph P_n obtained by means of the edge-adjacency matrix is*

$$P^e_{P_n}(\lambda) = \lambda P^e_{P_{n-1}}(\lambda) - P^e_{P_{n-2}}(\lambda).$$

Proof. The edge-adjacency matrix of P_n is

$$\mathrm{A}^e(P_n) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \ldots & 0 & 0 & 0 \\ . & . & . & . & . & \ldots & . & . & . \\ . & . & . & . & . & \ldots & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \end{bmatrix}$$

And characteristic polynomial of $\mathrm{A}^e(P_n)$ is

$$|\lambda.I - \mathrm{A}^e(P_n)| = \begin{vmatrix} \lambda & -1 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ -1 & \lambda & -1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & -1 & \lambda & -1 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & -1 & \lambda & -1 & ... & 0 & 0 & 0 \\ . & . & . & . & . & ... & . & . & . \\ . & . & . & . & . & ... & . & . & . \\ 0 & 0 & 0 & 0 & 0 & ... & -1 & \lambda & -1 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & -1 & \lambda \end{vmatrix}$$

If we calculate this determinant according to the first row, we get

$$\begin{aligned} |\lambda.I - \mathrm{A}^e(P_n)| &= \lambda P^e_{P_{n-1}}(\lambda) + 1 \begin{vmatrix} -1 & -1 & 0 & 0 & ... & 0 & 0 \\ 0 & \lambda & -1 & 0 & ... & 0 & 0 \\ 0 & -1 & \lambda & -1 & ... & 0 & 0 \\ 0 & 0 & -1 & \lambda & ... & 0 & 0 \\ . & . & . & . & & . & . \\ . & . & . & . & & . & . \\ . & . & . & . & & . & . \\ 0 & 0 & 0 & 0 & ... & \lambda & -1 \\ 0 & 0 & 0 & 0 & ... & -1 & \lambda \end{vmatrix} \\ &= \lambda P^e_{P_{n-1}}(\lambda) - P^e_{P_{n-2}}(\lambda) \end{aligned}$$

when we calculate this last determinant according to the first column.

□

The first few edge-characteristic polynomials corresponding to the path graphs can easily be obtained by means of the recurrence formula as

$$\begin{aligned} P^e_{P_3}(\lambda) &= \lambda^2 - 1 \\ P^e_{P_4}(\lambda) &= \lambda^3 - 2\lambda \\ P^e_{P_5}(\lambda) &= \lambda^4 - 3\lambda^2 - 1 \\ P^e_{P_6}(\lambda) &= \lambda^5 - 4\lambda^3 + 3\lambda. \end{aligned}$$

We can now obtain the formula for the edge-characteristic polynomial corresponding to the path graph P_n. We first need the following result:

Lemma 2.1. *For all positive integers k and n such that $k \leq n$, we have*

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

Proof. By the combinatorial properties of combination numbers, we have

$$\begin{aligned}\binom{n}{k}+\binom{n}{k-1} &= \frac{n!}{(n-k)!k!}+\frac{n!}{(n-k+1)!(k-1)!}\\ &= \frac{n!(n+1)}{(n-k+1)!k!}\\ &= \frac{(n+1)!}{(n+1-k)!k!}\\ &= \binom{n+1}{k}.\end{aligned}$$

□

Theorem 2.2.

$$P^e_{P_n}(\lambda)=\begin{cases}\sum_{l=0}^{\frac{n-1}{2}}(-1)^l\binom{n-1-l}{l}\lambda^{n-1-2l}, & n \text{ is odd}\\ \sum_{l=0}^{\frac{n-2}{2}}(-1)^l\binom{n-1-l}{l}\lambda^{n-1-2l}, & n \text{ is even.}\end{cases}$$

Proof. We shall use strong induction. Let n be odd.

$$\begin{aligned}P^e_{P_3} &= \lambda^2-1\\ &= (-1)^0\binom{2}{0}\lambda^2+(-1)^1\binom{1}{1}\lambda^0\\ &= \sum_{l=0}^{\frac{3-1}{2}}(-1)^l\binom{3-1-l}{l}\lambda^{3-1-2l}\end{aligned}$$

Let the statement be verified for $3,\ 4,\ 5,\ \cdots,\ n-1,\ n$. Then we want to prove that it is verified for $n+1$.

By the induction hypothesis, we know that

$$P^e_{P_n}=\sum_{l=0}^{\frac{n-1}{2}}(-1)^l\binom{n-1-l}{l}\lambda^{n-1-2l} \quad\text{and}\quad P^e_{P_{n-1}}=\sum_{l=0}^{\frac{n-3}{2}}(-1)^l\binom{n-2-l}{l}\lambda^{n-2-2l}.$$

So we get

$$\begin{aligned}
P^e_{P_{n+1}} &= \lambda P^e_{P_n}(\lambda) - P^e_{P_{n-1}}(\lambda)\\
&= \lambda\left(\sum_{l=0}^{\frac{n-1}{2}} (-1)^l \binom{n-1-l}{l}\lambda^{n-1-2l}\right) - \left(\sum_{l=0}^{\frac{n-3}{2}} (-1)^l \binom{n-2-l}{l}\lambda^{n-2-2l}\right)\\
&= \lambda\Big[(-1)^0\binom{n-1}{0}\lambda^{n-1} + (-1)^1\binom{n-2}{1}\lambda^{n-3} + (-1)^2\binom{n-3}{2}\lambda^{n-5} + (-1)^3\binom{n-4}{3}\lambda^{n-7}\\
&\quad + \cdots + (-1)^{\frac{n-1}{2}}\binom{n-1-\frac{n-1}{2}}{\frac{n-1}{2}}\lambda^0\Big] - \Big[(-1)^0\binom{n-2}{0}\lambda^{n-2} + (-1)^1\binom{n-3}{1}\lambda^{n-4}\\
&\quad + (-1)^2\binom{n-4}{2}\lambda^{n-6} + (-1)^3\binom{n-5}{3}\lambda^{n-8} + \cdots + (-1)^{\frac{n-3}{2}}\binom{n-2-\frac{n-3}{2}}{\frac{n-3}{2}}\lambda^1\Big]\\
&= (-1)^0\binom{n-1}{0}\lambda^n + (-1)^1\binom{n-1}{1}\lambda^{n-2} + \cdots + (-1)^{\frac{n-1}{2}}\binom{n-\frac{n-1}{2}}{\frac{n-1}{2}}\lambda\\
&= \sum_{l=0}^{\frac{n-1}{2}} (-1)^l\binom{n-l}{l}\lambda^{n-2l},
\end{aligned}$$

by Lemma 2.1.

Let now n be even.

$$\begin{aligned}
P^e_{P_4} &= \lambda^3 - 2\lambda\\
&= (-1)^0\binom{3}{0}\lambda^3 + (-1)^1\binom{2}{1}\lambda\\
&= \sum_{l=0}^{\frac{4-2}{2}} (-1)^l\binom{4-1-l}{l}\lambda^{4-1-2l}
\end{aligned}$$

Let the statement be verified for $3,\ 4,\ 5,\ \cdots,\ n-1,\ n$. Then we want to prove that it is verified for $n+1$.

By the induction hypothesis, we know that

$$P^e_{P_n} = \sum_{l=0}^{\frac{n-2}{2}} (-1)^l\binom{n-1-l}{l}\lambda^{n-1-2l} \quad \text{and} \quad P^e_{P_{n-1}} = \sum_{l=0}^{\frac{n-2}{2}} (-1)^l\binom{n-2-l}{l}\lambda^{n-2-2l}.$$

So we get

$$\begin{aligned}
P^e_{P_{n+1}} &= \lambda P^e_{P_n}(\lambda) - P^e_{P_{n-1}}(\lambda) \\
&= \lambda\left(\sum_{l=0}^{\frac{n-2}{2}} (-1)^l \binom{n-1-l}{l} \lambda^{n-1-2l}\right) - \left(\sum_{l=0}^{\frac{n-2}{2}} (-1)^l \binom{n-2-l}{l} \lambda^{n-2-2l}\right) \\
&= (-1)^0 \binom{n-1}{0} \lambda^n + (-1)^1 \binom{n-1}{1} \lambda^{n-2} + \cdots + (-1)^{\frac{n}{2}} \binom{\frac{n-2}{2}}{\frac{n-2}{2}} \lambda^0 \\
&= (-1)^0 \binom{n}{0} \lambda^n + (-1)^1 \binom{n-1}{1} \lambda^{n-2} + \cdots + (-1)^{\frac{n}{2}} \binom{\frac{n}{2}}{\frac{n}{2}} \lambda^0 \\
&= \sum_{l=0}^{\frac{n}{2}} (-1)^l \binom{n-l}{l} \lambda^{n-2l},
\end{aligned}$$

by Lemma 2.1.

□

Theorem 2.3. *The recurrence relation for the edge-characteristic polynomial of the cycle graph* C_n *obtained by means of the edge-adjacency matrix is*

$$P^e_{C_n}(\lambda) = \lambda P^e_{P_n}(\lambda) - P^e_{P_{n-1}}(\lambda) - 2.$$

Proof. First note that

$$\begin{aligned}
P^e_{C_3}(\lambda) &= \begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} \\
&= \lambda \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} + (-1)(-1) \begin{vmatrix} -1 & -1 \\ -1 & \lambda \end{vmatrix} + (-1) \begin{vmatrix} -1 & \lambda \\ -1 & -1 \end{vmatrix} \\
&= \lambda P^e_{P_3}(\lambda) + (-\lambda - 1) - (1 + \lambda) \\
&= \lambda P^e_{P_3}(\lambda) - 2\lambda - 2
\end{aligned}$$

and

$$\begin{aligned}
P^e_{C_4}(\lambda) &= \begin{vmatrix} \lambda & -1 & 0 & -1 \\ -1 & \lambda & -1 & 0 \\ 0 & -1 & \lambda & -1 \\ -1 & 0 & -1 & \lambda \end{vmatrix} \\
&= \lambda \begin{vmatrix} \lambda & -1 & 0 \\ -1 & \lambda & -1 \\ 0 & -1 & \lambda \end{vmatrix} + \begin{vmatrix} -1 & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} + \begin{vmatrix} -1 & \lambda & -1 \\ 0 & -1 & \lambda \\ -1 & 0 & -1 \end{vmatrix} \\
&= \lambda P^e_{P_4}(\lambda) - P^e_{P_3}(\lambda) + \begin{vmatrix} 0 & -1 \\ -1 & \lambda \end{vmatrix} - \begin{vmatrix} -1 & \lambda \\ 0 & -1 \end{vmatrix} - P^e_{P_3}(\lambda) \\
&= \lambda P^e_{P_4}(\lambda) - 2P^e_{P_3}(\lambda) - 1 - 1 \\
&= \lambda P^e_{P_4}(\lambda) - 2P^e_{P_3}(\lambda) - 2.
\end{aligned}$$

Similarly

$$P^e_{C_5}(\lambda) = \begin{vmatrix} \lambda & -1 & 0 & 0 & -1 \\ -1 & \lambda & -1 & 0 & 0 \\ 0 & -1 & \lambda & -1 & 0 \\ 0 & 0 & -1 & \lambda & -1 \\ -1 & 0 & 0 & -1 & \lambda \end{vmatrix}$$

$$\begin{aligned}
&= \lambda P^e_{P_5}(\lambda) + \begin{vmatrix} -1 & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & -1 & \lambda & -1 \\ -1 & 0 & -1 & \lambda \end{vmatrix} - \begin{vmatrix} -1 & \lambda & -1 & 0 \\ 0 & -1 & \lambda & -1 \\ 0 & 0 & -1 & \lambda \\ -1 & 0 & 0 & -1 \end{vmatrix} \\
&= \lambda P^e_{P_5}(\lambda) - P^e_{P_4}(\lambda) + \begin{vmatrix} 0 & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} + \begin{vmatrix} -1 & \lambda & -1 \\ 0 & -1 & \lambda \\ 0 & 0 & -1 \end{vmatrix} - P^e_{P_4}(\lambda) \\
&= \lambda P^e_{P_5}(\lambda) - 2P^e_{P_4}(\lambda) + \begin{vmatrix} 0 & -1 \\ -1 & \lambda \end{vmatrix} - 1 \\
&= \lambda P^e_{P_5}(\lambda) - 2P^e_{P_4}(\lambda) - 1 - 1 \\
&= \lambda P^e_{P_5}(\lambda) - 2P^e_{P_4}(\lambda) - 2.
\end{aligned}$$

Proceeding similarly, first we have to take determinant with respect to first row. Then we have $P^e_{P_n}(\lambda)$ and two determinants coming from the submatrix of $P^e_{C_n}(\lambda)$. And we take the determinant of last matrix with respect to first column and take determinant of former matrix with respect to first row. From the determinant of last matrix, we have an upper triangular matrix whose determinant is -1 and another matrix giving $-P^e_{P_{n-1}}(\lambda)$. From the determinant of the former matrix, we have $-P^e_{P_{n-1}}(\lambda)$ and a matrix. So if we take determinant of the matrix with respect to first row until reaching a number, finally we will have $\begin{vmatrix} 0 & -1 \\ -1 & \lambda \end{vmatrix}$. So it is equal to -1. The summation of terms gives

$$P^e_{C_n}(\lambda) = \lambda P^e_{P_n}(\lambda) - 2P^e_{P_{n-1}}(\lambda) - 2.$$

□

We know the formula of $P^e_{P_n}(\lambda)$. So if we use it, then we can have the formula for $P^e_{C_n}(\lambda)$, as follows:

Theorem 2.4. *The formula for the edge-characteristic polynomial of the cycle graph C_n is*

$$P^e_{C_n}(\lambda) = \lambda \sum_{l=0}^{\frac{n-2}{2}} (-1)^l \binom{n-1-l}{l} \lambda^{n-1-2l} - 2\sum_{l=0}^{\frac{n-2}{2}} (-1)^l \binom{n-2-l}{l} \lambda^{n-2-2l} - 2 \qquad \textit{if } n \textit{ is even},$$

and

$$P^e_{C_n}(\lambda) = \lambda \sum_{l=0}^{\frac{n-1}{2}} (-1)^l \binom{n-1-l}{l} \lambda^{n-1-2l} - 2\sum_{l=0}^{\frac{n-3}{2}} (-1)^l \binom{n-2-l}{l} \lambda^{n-2-2l} - 2 \qquad \textit{if } n \textit{ is odd}.$$

The following two results prove that the sum of the coefficients of $P^e_{P_n}$ and $P^e_{C_n}$ have a special form in modulo 6:

Theorem 2.5.

$$P^e_{P_n}(1) = \begin{cases} 0, & n \equiv 0\ (6) \textit{ and } n \equiv 3\ (6) \\ 1, & n \equiv 1\ (6) \textit{ and } n \equiv 2\ (6) \\ -1, & n \equiv 4\ (6) \textit{ and } n \equiv 5\ (6) \end{cases}$$

Theorem 2.6.

$$P^e_{C_n}(1) = \begin{cases} 0, & n \equiv 0\ (6) \\ -1, & n \equiv 1\ (6) \textit{ and } n \equiv 5\ (6) \\ -3, & n \equiv 2\ (6) \textit{ and } n \equiv 4\ (6) \\ -4, & n \equiv 3\ (6) \end{cases}$$

Theorem 2.7. *The recurrence formula for the edge-characteristic polynomial of the star graph S_n is*

$$P^e_{S_n}(\lambda) = \lambda P^e_{S_{n-1}}(\lambda) - (n-2)(\lambda+1)^{n-3}.$$

Proof. Edge adjacency matrix of the star graph having n vertices can be obtained in the form

$$\mathrm{A}^e(S_n) = \begin{bmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}_{(n-1)\times(n-1)} .$$

Characteristic polynomial of $\mathrm{A}^e(S_n)$ is

$$|\lambda.I - \mathrm{A}^e(S_n)| = \begin{vmatrix} \lambda & -1 & \cdots & -1 & -1 \\ -1 & \lambda & \cdots & -1 & -1 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ -1 & -1 & \cdots & \lambda & -1 \\ -1 & -1 & \cdots & -1 & \lambda \end{vmatrix}_{(n-1)\times(n-1)} .$$

If we use determinant rule with respect to first row then it will give three determinants of size $(n-2)\times(n-2)$

$$P^e_{S_n}(\lambda) = \lambda P^e_{S_{n-1}} + (-1)^{1+2}.(-1) \begin{vmatrix} -1 & -1 & \cdots & -1 & -1 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ -1 & -1 & \cdots & \lambda & -1 \\ -1 & -1 & \cdots & -1 & \lambda \end{vmatrix}$$

$$+(-1)^{1+3}.(-1) \begin{vmatrix} -1 & \lambda & \cdots & -1 & -1 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ -1 & -1 & \cdots & \lambda & -1 \\ -1 & -1 & \cdots & -1 & \lambda \end{vmatrix} + \cdots + (-1)^{1+n}.(-1) \begin{vmatrix} -1 & \lambda & \cdots & -1 & -1 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ -1 & -1 & \cdots & -1 & \lambda \\ -1 & -1 & \cdots & -1 & -1 \end{vmatrix}$$

and then if we apply elemantary row operations to the determinants

$$\begin{vmatrix} -1 & -1 & \cdots & -1 & -1 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ -1 & -1 & \cdots & \lambda & -1 \\ -1 & -1 & \cdots & -1 & \lambda \end{vmatrix}, \begin{vmatrix} -1 & \lambda & \cdots & -1 & -1 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ -1 & -1 & \cdots & \lambda & -1 \\ -1 & -1 & \cdots & -1 & \lambda \end{vmatrix}, \cdots, \begin{vmatrix} -1 & \lambda & \cdots & -1 & -1 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ -1 & -1 & \cdots & -1 & \lambda \\ -1 & -1 & \cdots & -1 & -1 \end{vmatrix},$$

we get

$$P^e_{S_n}(\lambda) = \lambda P^e_{S_{n-1}} + (n-2)\begin{vmatrix} -1 & -1 & \cdots & -1 & -1 \\ -1 & \lambda & \cdots & -1 & -1 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ -1 & -1 & \cdots & \lambda & -1 \\ -1 & -1 & \cdots & -1 & \lambda \end{vmatrix}_{(n-2)\times(n-2)}$$

as all the determinants are the same. In this last matrix, if we use the row operations

$$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1,\ \cdots,\ R_n \to R_n - R_1,$$

then it will be in the form

$$\begin{vmatrix} -1 & -1 & \cdots & -1 & -1 \\ 0 & \lambda+1 & \cdots & -1 & -1 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \lambda+1 & -1 \\ 0 & 0 & \cdots & 0 & \lambda+1 \end{vmatrix}_{(n-2)\times(n-2)}.$$

So it is equal to $(-1)(\lambda+1)^{n-3}$. Hence

$$P^e_{S_n}(\lambda) = \lambda P^e_{S_{n-1}}(\lambda) - (n-2)(\lambda+1)^{n-3}.$$

□

Theorem 2.8. *The edge-characteristic polynomial of the star graph S_n is*

$$P^e_{S_n}(\lambda) = [\lambda - (n-2)](\lambda+1)^{(n-2)}.$$

Proof. Edge adjacency matrix of star graph with n vertices is in the form

$$P^e_{S_n}(\lambda) = \begin{vmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{vmatrix}_{(n-1)\times(n-1)}.$$

Characteristic polynomial of $A^e(S_n)$ is

$$|\lambda.I - \mathrm{A}^e(S_n)| = \begin{vmatrix} \lambda & -1 & \cdots & -1 & -1 \\ -1 & \lambda & \cdots & -1 & -1 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ -1 & -1 & \cdots & \lambda & -1 \\ -1 & -1 & \cdots & -1 & \lambda \end{vmatrix}_{(n-1)\times(n-1)} .$$

Let us use elementary operations which are firstly $R_1 \to R_1 + R_2 + R_3 + \cdots + R_{n-1}$, then the matrix will be

$$\begin{vmatrix} \lambda-(n-2) & \lambda-(n-2) & \cdots & \lambda-(n-2) & \lambda-(n-2) \\ -1 & \lambda & \cdots & -1 & -1 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ -1 & -1 & \cdots & \lambda & -1 \\ -1 & -1 & \cdots & -1 & \lambda \end{vmatrix} .$$

After that from the determinant property, it is equal to

$$[\lambda-(n-2)] \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & \lambda & \cdots & -1 & -1 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ -1 & -1 & \cdots & \lambda & -1 \\ -1 & -1 & \cdots & -1 & \lambda \end{vmatrix}_{(n-1)\times(n-1)}$$

and if we apply $R_2 \to R_2 + R_1,\ R_3 \to R_3 + R_1,\ \cdots,\ R_{n-1} \to R_{n-1} + R_1$, then we have

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & \lambda+1 & \cdots & 0 & 0 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \lambda+1 & 0 \\ 0 & 0 & \cdots & 0 & \lambda+1 \end{vmatrix}_{(n-1)\times(n-1)} = (\lambda+1)^{(n-2)}.$$

Hence

$$P^e_{S_n}(\lambda) = [\lambda-(n-2)](\lambda+1)^{(n-2)}.$$

□

The following results can be obtained similarly to the above results:

Theorem 2.9. *The recurrence relation for the edge-characteristic polynomial of the tadpole graph $T_{r,s}$ obtained by means of the edge-adjacency matrix is*

$$P^e_{T_{r,s}}(\lambda) = \begin{cases} \lambda P^e_{T_{r,s-1}}(\lambda) - P^e_{T_{r,s-2}}(\lambda), & s \neq 1,2 \\ \lambda P^e_{T_{r,s-1}}(\lambda) - P^e_{C_r}(\lambda), & s = 2 \\ \lambda P^e_{C_r}(\lambda) - 2P^e_{P_r}(\lambda) - 2P^e_{P_{r-1}}(\lambda) - 2, & s = 1 \end{cases}$$

Theorem 2.10. *The edge-characteristic polynomial of the complete graph K_n is*

$$P^e_{K_n}(\lambda) = \begin{cases} (\lambda-2)(\lambda+1)^2, & n = 3 \\ (\lambda - 2(n-2))(\lambda - (n-4))^{n-1}(\lambda+2)^{\binom{n}{2}-n}, & n \neq 3. \end{cases}$$

Theorem 2.11. *The edge-characteristic polynomial of the complete bipartite graph $K_{r,s}$ is*

$$P^e_{K_{r,s}}(\lambda) = (\lambda - s - r + 2)(\lambda - s + 2)^{r-1}(\lambda - r + 2)^{s-1}(\lambda+2)^{(r-1)(s-1)}$$

The following results giving some relations between edge-adjacency and vertex-adjacency polynomials can be obtained directly from the above results and their proofs will be omitted:

Theorem 2.12. $P^e_{P_n}(\lambda) = P^v_{P_{n-1}}(\lambda)$.

Theorem 2.13. $P^e_{C_n}(\lambda) = P^v_{C_n}(\lambda)$.

Theorem 2.14. $P^e_{K_{r,1}}(\lambda) = P^e_{S_{r+1}}(\lambda)$.

Theorem 2.15. $P^e_{S_n}(\lambda) = P^v_{K_{n-1}}(\lambda)$.

3 Incidency matrices and polynomials

Up to now, we have studied the vertex-characteristic and edge-characteristic polynomials. As well known, the former ones play a very important role in spectral graph theory in calculation of the energy of a given graph. Finally, we study the incidency polynomials. Recall that if $e = uv$ is an edge of a graph G, then e is said to be incident to the vertices u and v in G. The incidency matrix $A^i(G)$ is defined as an $m \times n$ matrix. $\mathrm{A}^i(G) = [a_{ij}]_{m\times n}$ determined by the incidency of edges and vertices as follows:

$$a_{ij} = \begin{cases} 1, & \text{if the vertex } v_i \text{ is incident to the edge } e_j \\ 0, & \text{otherwise.} \end{cases}$$

Now we shall calculate the formulae and recurrence relations for the incidency polynomials.

As the incidency matrix for a given graph G need not be a square matrix, it is not possible to calculate the incidency polynomial of all graphs. The following are few examples to those graphs whose incidency matrix is a square matrix and hence whose incidency polynomials can be calculated:

Lemma 3.1. *If we divide a matrix into four block matrices*

$$\left[\begin{array}{c|c} A & 0 \\ \hline C & B \end{array}\right]$$

where A and B are square matrices, the determinant of this matrix is equal to $|A|\,|B|$.

By means of this lemma, we can obtain the recurrence relation for tadpole graphs:

Theorem 3.1. *The incidency polynomial of the tadpole graph $T_{r,s}$ is*

$$I_{T_{r,s}}(\lambda) = (\lambda - 1)^s I_{C_r}(\lambda).$$

Proof. Incidency matrix of the tadpole grapf $T_{r,s}$ is in the form

$$a_{ij} = \begin{cases} 1, & \text{if } i = j \\ 1, & \text{if } i - j = 1 \\ 1, & \text{if } i = s+1, j = r+s \\ 0, & \text{otherwise.} \end{cases}$$

Let us divide the matrix $A^i(T_{r,s})$ into the block matrices as below:

$$A^i(T_{r,s}) = \left[\begin{array}{cccccccc|ccccc} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 & & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 & & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & \cdots & . & . & & . & . & . & . & . \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 & 1 & & 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & 0 & 0 & \cdots & 0 & 0 \\ . & . & . & . & \cdots & . & . & & . & . & \cdots & . & . \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & 0 & 0 & \cdots & 1 & 1 \end{array}\right]$$

and

$$|\lambda.I - \mathrm{A}^i(T_{r,s})| = \left|\begin{array}{ccccccc|ccccc}
\lambda-1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & \lambda-1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & \lambda-1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & . & . & . & \cdots & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & \cdots & -1 & \lambda-1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \cdots & 0 & -1 & \lambda-1 & 0 & \cdots & 0 & -1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \lambda-1 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
. & . & . & . & \cdots & . & . & . & . & \cdots & . & . \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \lambda-1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -1 & \lambda-1
\end{array}\right|.$$

We have seen that the determinant of upper-left block matrix is $(\lambda-1)^s$ and the determinant of lower-right block matrix is $I_{C_r}(\lambda)$. Hence from Lemma 3.1,

$$\left|\lambda I - A\right| = (\lambda-1)^s I_{C_r}(\lambda)$$

□

The following two results can be proven similarly:

Theorem 3.2. *The incidency polynomial of the cycle graph C_n is*

$$I_{C_n}(\lambda) = (\lambda-1)^n - 1$$

Theorem 3.3. *The incidency polynomial of the complete bipartite graph $K_{2,2}$ is*

$$I_{K_{2,2}}(\lambda) = (\lambda-2)(\lambda-1)\lambda^2.$$

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